B.Sc part-II (H), paper-III Tangents and normals. 25-07-2020

Problems based on pedal eqn of a Cartesian curve

Q.1. In the equilateral hyperbola $x^2-y^2=a^2$, prove that $pr=a^2$.

Soln: $F(x,y)\equiv x^2-y^2-a^2=0$

Making the equation homogeneous, we have

$\cancel{F\equiv x^2-y^2-a^2z^2=0}$

$F\equiv x^2-y^2-a^2z^2=0$, where $z=1$

$\therefore F_z=-2a^2z=-2a^2,\ F_x=2x,\ F_y=-2y$

$\therefore p^2=\dfrac{F_z^2}{F_x^2+F_y^2}=\dfrac{a^4}{x^2+y^2}=\dfrac{a^4}{2x^2-a^2}$ —— (I)

And $x^2=a^2+y^2$ —— (II)

$r^2=x^2+y^2$ —— (III)

$y^2=r^2-x^2$
$x^2=a^2+r^2-x^2$
$\Rightarrow x^2+x^2=a^2+r^2$
$\Rightarrow 2x^2=a^2+r^2$
$\Rightarrow 2x^2-a^2=r^2$

From (I) and (III).

$2x^2-a^2=r^2$

$\therefore p^2=\dfrac{a^4}{r^2}\Rightarrow pr=a^2$. [from (I)]

Proved.

Q.2. Show that the pedal equation of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with regard to its centre is $\dfrac{a^2b^2}{p^2}=a^2+b^2-r^2$

Soln: $\because F(x,y)\equiv\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$

or, $F\equiv\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-z^2=0$ where $z=1$.

$\therefore F_z=-2z=-2,\ F_x=\dfrac{2x}{a^2},\ F_y=\dfrac{2y}{b^2}$

$\therefore p^2=\dfrac{F_z^2}{F_x^2+F_y^2}=\dfrac{1}{\frac{x^2}{a^4}+\frac{y^4}{b^4}}$ or, $\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}-\dfrac{1}{p^2}=0$ —— (I)

Also, $x^2+y^2-r^2=0$ —— (II) and $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$ —— (III)

Eliminating x^2 and y^2 from (I), (III) and II, we get

$$\begin{vmatrix}\frac{1}{a^2} & \frac{1}{b^4} & \frac{1}{p^2}\\ 1 & 1 & r^2\\ \frac{1}{a^2} & \frac{1}{b^2} & 1\end{vmatrix}=0$$ —— (IV)

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$$\Rightarrow \begin{vmatrix} a^2 & b^2 & p^2 \\ a^4 & b^4 & p^2r^2 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Which is the required pedal equation.

Expanding this determinant, we get

$$p^2(a^4-b^4) - p^2r^2(a^2-b^2) + 1\cdot a^2b^2(b^2-a^2) = 0$$

$$\Rightarrow p^2(a^2+b^2) - p^2r^2 - a^2b^2 = 0$$

$$\Rightarrow \frac{a^2b^2}{p^2} = a^2+b^2-r^2$$

Q.3 Show that the pedal equation of the astroid.

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \text{ is } r^2+3p^2 = a^2$$

Soln: We have, $p^2 = \dfrac{(xf_x + yf_y)^2}{f_x^2 + f_y^2}$

$$= \frac{\left[x\left(\frac{2}{3}x^{-\frac{1}{3}}\right) + y\left(\frac{2}{3}y^{-\frac{1}{3}}\right)\right]^2}{\left(\frac{2}{3}x^{-\frac{1}{3}}\right)^2 + \left(\frac{2}{3}y^{-\frac{1}{3}}\right)^2} = \frac{\frac{4}{9}\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^2}{\frac{4}{9}\left(x^{-\frac{2}{3}}+y^{-\frac{2}{3}}\right)} = \frac{a^{\frac{4}{3}}}{x^{-\frac{2}{3}}+y^{-\frac{2}{3}}}$$

$$= \frac{a^{\frac{4}{3}} \times x^{\frac{2}{3}}y^{\frac{2}{3}}}{x^{\frac{2}{3}}+y^{\frac{2}{3}}} = \frac{a^{\frac{4}{3}}x^{\frac{2}{3}}y^{\frac{2}{3}}}{a^{\frac{2}{3}}} = a^{\frac{2}{3}}x^{\frac{2}{3}}y^{\frac{2}{3}}$$

$$\therefore 3p^2 = 3a^{\frac{2}{3}}x^{\frac{2}{3}}y^{\frac{2}{3}}$$

Now, $a^2-r^2 = a^2-(x^2+y^2)$

$$= a^2 - \left[\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^3 - 3x^{\frac{2}{3}}y^{\frac{2}{3}}\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)\right]$$

$$= a^2 - \left[a^2 - 3a^{\frac{2}{3}}x^{\frac{2}{3}}y^{\frac{2}{3}}\right] = 3a^{\frac{2}{3}}x^{\frac{2}{3}}y^{\frac{2}{3}}$$

$$\therefore 3p^2 = a^2-r^2.$$

$$\therefore r^2+3p^2 = a^2$$

Proved.